9.3 (continued)

Boundary-value Problem (BVP)

$x'' + ax = f(t)$ $\qquad \underbrace{x(0)=0, \quad x(L)=0}_{\text{not at same place like initial-value problem}}$ $\qquad$ Dirichlet condition (values at ends)

$x'(0)=0, \quad x'(L)=0$ $\qquad$ Neumann condition (rates at ends)

if $f(t)=0$, $\underbrace{x'' + ax = 0}$ $\qquad x(0) = x(L) = 0$

$$x(t) = A\cos(\sqrt{a}\,t) + B\sin(\sqrt{a}\,t)$$

$x(0)=0 \rightarrow 0 = A$

$x(L)=0 \rightarrow 0 = B\sin(\sqrt{a}\,L)$ $\qquad B \neq 0$ (otherwise $x=0$ for all t) trivial solution

$\sin(\sqrt{a}L) = 0 \qquad \sqrt{a}L = n\pi \qquad n = 1, 2, 3, \dots$

this means, for each $n = 1, 2, 3, \dots$

there is one fundamental solution :

from $x(t) = \underset{0}{A}\cos(\sqrt{a}t) + \underset{\neq 0}{B}\sin(\sqrt{a}t)$

$$X_n = \sin\Big(\underbrace{\frac{n\pi}{L}}_{\sqrt{a}}t\Big) \qquad n = 1, 2, 3, \dots$$

there are infinitely-many n's, so the general solution is a linear combination of ALL :

$$\boxed{x(t) = b_1 \sin\left(\frac{\pi}{L}t\right) + b_2 \sin\left(\frac{2\pi}{L}t\right) + b_3 \sin\left(\frac{3\pi}{L}t\right) + \cdots}$$

Sine series w/ half period L

$x'' + ax = 0$ $\quad \underbrace{x(0) = x(L) = 0}_{\text{each } \sin(\frac{n\pi}{L}t) \text{ satisfies the boundary conditions}}$ is solved by the sine series above

if we had $x'(0) = x'(L) = 0$, we would get a cosine series

now let's look at $f(t) \neq 0$

$$\underbrace{x'' + ax}_{\substack{\text{sine series}\\ \text{if } f(t)=0}} = f(t) \qquad x(0) = x(L) = 0$$

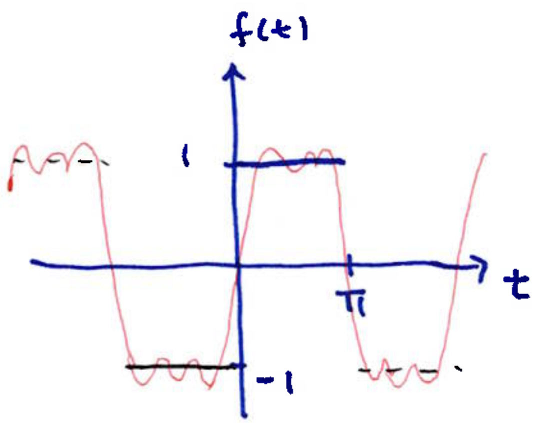
Dirichlet condition
(HW: Neumann)

we need to add the appropriate extensions (even or odd)
to $f(t)$ to make it periodic
such that it satisfies the boundary conditions

example $x'' + 2x = 1$ $x(0)=0$, $x(\pi)=0$

tells us $L=\pi$

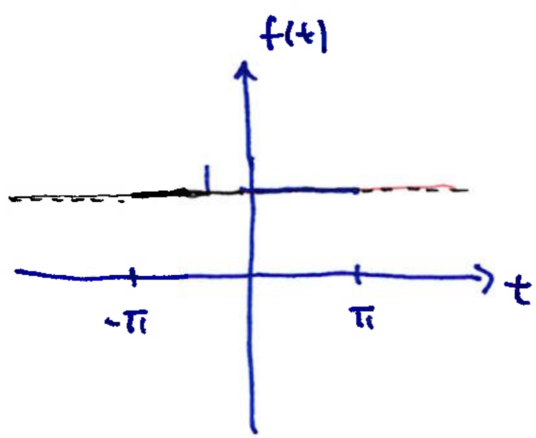
$f(t)=1$ we need to add even or odd extensions so it satisfies BC's and is periodic



w/ odd extensions

its Fourier series (in red)

notice it satisfies ~~x(0)~~ $f(0)=f(\pi)=0$ (BC's)



w/ even extensions

its Fourier series (in red)

does <u>NOT</u> satisfy the BC's

BUT, it DOES satisfy if

$x'(0)=x'(\pi)=0$

so, we want <u>odd</u> extensions in this example for $f(t) = 1$ $\quad L = \pi$

$$f(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases} \qquad \text{period } 2\pi$$

we write out a Fourier series for this

$\vdots$

$$f(t) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right]\sin(nt) \qquad (\text{sine series})$$

back to $\underbrace{x''+2x}_{\substack{\text{if right side is 0} \\ \text{Sine series}}} = 1 \qquad x(0) = x(\pi) = 0$

express $x(t)$ as a sine series w/ unknown coefficients w/ $L = \pi$

$$x(t) = \sum_{n=1}^{\infty} B_n \sin(nt) \qquad \text{sub into } x''+2x = 1$$

$$x'(t) = \sum_{n=1}^{\infty} n B_n \cos(nt)$$

$$x''(t) = \sum_{n=1}^{\infty} -n^2 B_n \sin(nt)$$

Sub into $x''+2x=1$ w/ 1 replaced by its Fourier Series

$$\sum_{n=1}^{\infty} -n^2 B_n \sin(nt) + 2\sum_{n=1}^{\infty} B_n \sin(nt) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right]\sin(nt)$$

for each $n=1,2,3,\ldots$ equate coefficients of $\sin(nt)$

$$-n^2 B_n + 2B_n = \frac{2}{n\pi}\left[1-(-1)^n\right]$$

$$B_n = \frac{2\left[1-(-1)^n\right]}{n\pi(2-n^2)} \qquad n=1,2,3,\ldots$$

solution: $x(t) = \sum_{n=1}^{\infty} B_n \sin(nt)$

$$= \sum_{n=1}^{\infty} \frac{2\left[1-(-1)^n\right]}{n\pi(2-n^2)}\sin(nt)$$

$$= \frac{4}{\pi}\sin(t) - \frac{4}{21\pi}\sin(3t) - \frac{4}{105\pi}\sin(5t) - \cdots$$